Piston theory applied to strong shocks and unsteady flow

By JOSEPH L. RAYMOND
Aero-Astronautics Department, The RAND Corporation, Santa Monica, California

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The utility of piston theory as a means of solving isentropic two-dimensional aerodynamic problems has been aptly demonstrated by many writers in recent years. The present treatment removes the restriction of isentropic flows, extending the applicability of piston theory to flows with strong shocks. Sample calculations for a thin biconvex airfoil are carried out in which the local flow is assumed to be isentropic and non-isentropic. Comparison of the result is made with that of the shock expansion theory of Cole, Gazley & Williams (1956).

1. Introduction

Several papers have been published in recent years which deal with various problems of isentropic shock-free flows; for example, Lighthill (1953), Ashley & Zartarian (1956), Landahl (1957), and Chawla (1958). The condition of isentropy in essence defines a reasonable upper limit for the piston speed of the order of the speed of sound, i.e. $w/a_\infty \leqslant 1$, where w is the local velocity normal to the main undisturbed flow and a_∞ is the undisturbed speed of sound. However, according to Hayes (1947, 1957) this is a restriction which does not seem always justifiable.

In the development of the piston analogy there is no effect of the piston speed which limits its validity. The local Mach number, M, must be large, and since Hayes (1947) shows that $M = O(\delta^{-1})$, where δ is local slope, it is only necessary that δ be sufficiently small. In general, the error in pressure coefficients or other parameters of interest are of the order δ^2 for all Mach numbers, as pointed out by Van Dyke (1954). This again requires that the local slopes be small in order to limit errors in the pressure coefficient, but does not necessarily limit the piston speed, $K_\infty \equiv M_\infty \delta$.

Assuming that the piston theory is valid for all speeds including those corresponding with strong shock flow, where $K_\infty > 1$, then there is an extension of the works on piston theory to flows with strong steady and unsteady shocks. Hence, we seek a relationship between the local piston speed, w, and the local pressure, p, for strong shock flows, since the corresponding pressure relationship for isentropic flows is well known.

2. The strong-shock pressure function

Previously, Raymond & Williams (1957) found an empirical relationship for the reduced pressure coefficients, C_p/δ^2, based on the hypersonic small-disturbance theory of Van Dyke (1954) or the piston theory of Cole *et al.* (1956). It is

$$\bar{C}_{p_c} + \bar{C}_{p_e} = \gamma + 1, \tag{1}$$

where the coefficients on the left-hand side refer to compression and expansion respectively. The isentropic expression for the reduced expansion pressure coefficient is

$$\bar{C}_{p_e} = \frac{2}{\gamma K_\infty^2}\left[\left(1-\frac{\gamma-1}{2}K_\infty\right)^{2\gamma/\gamma-1}-1\right]. \tag{2a}$$

For $K_\infty \geqslant 1\cdot4$, this comes within 4 % of the value given by

$$\bar{C}_{p_e} = -\frac{2}{\gamma K_\infty^2}. \tag{2b}$$

Using equation (2*b*) for the sake of simplicity, equation (1) becomes

$$\bar{C}_{p_c} = \gamma+1+\frac{2}{\gamma K_\infty^2}. \tag{3}$$

Hypersonic flows over small local slopes are typified by the head shock lying close to and almost parallel to the body. In the case of a two-dimensional body, a component of flow, u_n, is generated normal to the local surface and is related to the shock-induced normal flow component u_s, by

$$u_s = \frac{u_n}{\cos(\theta_w-\delta)}. \tag{4}$$

For $\theta_w = O(\delta) \ll 1$, we find, from equation (4), that $u_s \simeq u_n$, and $u_n \simeq U_\infty \delta$ from the usual small-angle approximations applied to the body surface.

We now proceed to some of the inferences that can be made with the use of the small-angle approximations. Recalling the previous remarks on the limitation of piston speed, $w \leqslant a_\infty$, we seek now the relationship between w and p.

From equation (3) we get

$$\frac{p_c}{p_\infty} = \frac{\gamma K_\infty^2}{2}(\gamma+1)+2, \tag{5a}$$

but we see that $K_\infty = U_\infty\delta/a_\infty \simeq w/a_\infty$ within the small-angle approximation, hence

$$\frac{p_c}{p_\infty} = \frac{\gamma(\gamma+1)}{2}\left(\frac{w}{a_\infty}\right)^2+2. \tag{5b}$$

Equation (5*b*) was predicted theoretically by the piston analysis of Cole *et al.*, except that they were only able to show that the second term on the right-hand side was $O(1)$.

3. Application to a biconvex airfoil

It is relatively simple to calculate w/a_∞ as a function of the time (or the x-co-ordinate) and then use the appropriate piston equation for pressure coefficient depending upon whether the flow is isentropic or not. The following developments were considered by Lighthill (1953), and to a large extent we shall make use of his nomenclature.

Let y be the co-ordinate normal to the body and positive away from the centre line, then

$$y = Y_s(x) \pm Y_i(x,t) \text{ on the } \begin{Bmatrix}\text{lower}\\ \text{upper}\end{Bmatrix} \text{ surface,} \tag{6}$$

where Y_s is the equation of the profile shape and Y_i is the unsteady y-displacement. Differentiating equation (6), we get

$$w = w_s(x) \pm w_i(x,t) \text{ on the } \begin{Bmatrix}\text{lower}\\ \text{upper}\end{Bmatrix} \text{ surface.} \tag{7}$$

In general, for one-dimensional flow

$$w = \frac{\partial Y}{\partial t} + U_\infty \frac{\partial Y}{\partial x}, \tag{8}$$

so that if we take a thin biconvex airfoil as an example, we get

$$Y_s = 2\tau[x-(x^2/c)], \tag{9}$$

where $\tau = t/c$ is the thickness-to-chord ratio. Then from equations (8) and (9),

$$w_s = 2U_\infty \tau[1-(2x/c)]. \tag{10}$$

Let

$$Y_i = \alpha(t)\,(x-x_0), \tag{11}$$

then from equation (8)

$$w_i = \dot{\alpha}(x-x_0)+U_\infty \alpha. \tag{11a}$$

Let $\alpha = \alpha_0 e^{i\omega t}$, then equation (11$a$) becomes

$$w_i = \alpha_0 e^{i\omega t}\{i\omega(x-x_0)+U_\infty\}, \tag{12}$$

and adding equations (10) and (12) results in

$$\frac{w}{a_\infty} = \frac{w_s+w_i}{a_\infty} = 2M_\infty \tau\left[1-\frac{2x}{c}\right] \pm \alpha_0 e^{i\omega t}\left[\frac{i\omega(x-x_0)}{a_\infty}+M_\infty\right]. \tag{13}$$

Taking the real parts, we obtain

$$\frac{w}{a_\infty} = 2M_\infty \tau\left[1-\frac{2x}{c}\right] \pm \alpha_0 \left[M_\infty \cos\omega t - \frac{\omega(x-x_0)}{a_\infty}\sin\omega t\right]. \tag{14}$$

The last equation is not in its most convenient form from the conventional point of view. Letting $k = \omega c/2U_\infty$, equation (14) becomes

$$\frac{w}{a_\infty} = 2M_\infty \tau\left[1-\frac{2x}{c}\right] \pm \alpha_0 M_\infty\left[\cos\frac{2kx}{c}+\frac{2k}{c}(x_0-x)\sin\frac{2kx}{c}\right], \tag{15}$$

where $t = x/U_\infty$. This is essentially a statement of Hayes's piston transformation.

4. Force and moment calculation

For the biconvex airfoil thus selected, we calculate the force and moment coefficients:

$$C_N = \frac{2}{\gamma M_\infty^2}\int_0^1 \frac{(p_l-p_u)}{p_\infty}\frac{dx}{c}, \tag{16}$$

and

$$-C_M = \frac{2}{\gamma M_\infty^2}\int_0^1 \frac{(p_l-p_u)}{p_\infty}\frac{(x-x_0)}{c}\frac{dx}{c}, \tag{17}$$

where p_l and p_u are the local pressures on the lower and supper surfaces, respectively.

The pressure relationship for isentropic flow is

$$\frac{p}{p_\infty} = \left(1+\frac{\gamma-1}{2}\frac{w}{a_\infty}\right)^{2\gamma/\gamma-1}. \tag{18}$$

Now it is clear that some rules must be formulated for the use of equations (5*b*) and (18) in terms of piston Mach number. To help resolve this question in the range of piston Mach numbers $1{\cdot}0 \leqslant w/a_\infty \leqslant 1{\cdot}4$, the compression-pressure ratio was calculated based on both (5*b*) (valid for $w/a_\infty \geqslant 1{\cdot}4$ in the present approximation) and the corresponding equation valid in the range $1{\cdot}0 \leqslant w/a_\infty \leqslant 1{\cdot}4$,

$$\frac{p_c}{p_\infty} = \frac{\gamma(\gamma+1)}{2}\left(\frac{w}{a_\infty}\right)^2 + 2 - \left(1 - \frac{\gamma-1}{2}\frac{w}{a_\infty}\right)^{2\gamma/\gamma-1}. \tag{19}$$

(Equation (19) is based on equations (1) and (2*a*).) Both pressure ratios were compared with the isentropic value given by equation (18) for $w/a_\infty = 1{\cdot}0$. It was found that equations (5*b*) and (19) result in pressure ratios which are 3 % high and 3 % low, respectively, compared to equation (18). Consequently, since there appears to be no serious error in using equation (5*b*) over equation (19), the former was used because of simplicity in the example which follows. Clearly, some idealization is embraced here since aerodynamic flows do not transform abruptly from isentropic to strong shock flows as the above usage implies.

Suggested rules based on piston Mach numbers are

$\dfrac{w}{a_\infty} \leqslant \dfrac{-2}{\gamma-1}$,	$\dfrac{p}{p_\infty} = 0$	Isentropic flow
$\dfrac{-2}{\gamma-1} < \dfrac{w}{a_\infty} \leqslant 1$,	Equation (18)	
$\dfrac{w}{a_\infty} > 1$,	Equation (5*b*)	Strong-shock flow

5. Results

The biconvex airfoil of equation (9) with $\tau = 0{\cdot}025$, $\alpha_0 = 0{\cdot}1047$, $M_\infty = 10$, and $x_0/c = 0{\cdot}50$ was selected as an example calculation. The results are:

k	C_N	C_M
0	0·0521	0·00438
0·01	0·0521	0·00438
0·10	0·0516	0·00445
$\pi/4$	0·0339	0·00996
$\pi/2$	0·0252	0·01661
π	0·0699	0·00749

It will be noted that $M_\infty \tau$ and $M_\infty \alpha_0$ appear only in combination in equation (15); hence the calculations apply over a wide range of M_∞, τ and α_0.

A point of comparison was made for C_N at $k = 0$, with the results of the shock-expansion theory by Cole *et al.*, and the present value was found to be high by about 4·5 %.

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